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'Duality twisted' boundary conditions in *n*-state Potts models

G Schütz

Department of Nuclear Physics, Weizmann Institute, Rehovot, Israel

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Abstract. We discuss a new class of toroidal boundary conditions for one-dimensional quantum Hamiltonians with S_n symmetry which are related to two-dimensional n-state Potts models in the extreme anisotropic Hamiltonian limit. At their self-dual point (a point where a second-order phase transition occurs for $n=2, 3, 4$) the duality transformation is shown *to be an additional symmetry giving rise to a new class of 'duality twisted' toroidal boundary* conditions. The corresponding Hamiltonians are given in terms of generators of the periodic Temperley–Lieb algebra with an odd number of generators. We discuss as an example the critical Ising model. Here the complete spectrum for the new boundary conditions can be obtained from a projection mechanism in the spin-1/2 XXZ Heisenberg chain.

For a long time it has been known how to construct one-dimensional n-state quantum chains defined by a Hamiltonian $H(\lambda)$ from the transfer matrices of two-dimensional spin models defined on an Abelian group $\mathscr{A}$ of finite order n. This is achieved by taking an appropriate extreme anisotropic limit of the coupling constants in space and (euclidean) time direction [1, 2] in such a way that the parameter λ represents the inverse temperature of the corresponding two-dimensional model. $H(\lambda)$ is symmetric under a (discrete) symmetry group $\mathscr{G}$ of order $p \geqslant n$ of H (containing $\mathscr{A}$ as a subgroup). $\mathscr{G}$ depends on the choice of coupling constants to generalized magnetic fields which are other parameters of these models. Such models have been the object of extensive studies, well known examples include the Potts quantum chain corresponding the two-dimensional n-state Potts models [3] symmetric under the symmetric group S_n (see below), the n-state chiral Potts model [4] symmetric under the cyclic group Z_n or the Zamolodchikov Fateev quantum chain symmetric under the dihedral group D_n [5]. Many of these models are self-dual in the sense that the spectrum of a finite chain with N sites verifies for judiciously chosen boundary conditions B, B' the relation $E_S^B(\lambda) = \lambda E_S^{B'}(1/\lambda)$. ($E_Y^X(\lambda)$ represents suitably chosen subsets Y of eigenvalues of $H(\lambda)$ acting on a chain of N sites with some boundary condition denoted by X.)

It has been realized that for such Hamiltonians exist p different types of toroidal boundary conditions, each type corresponding to one of the p different elements $u \in \mathscr{G}$ [6]. This means that the Hamiltonian $H(\lambda)$ commutes with a generalized translation operator T_u which has the form $T_u = T \cdot u_N$ where T is the translation operator for periodic boundary conditions and u_N is an element of $\mathscr{G}$ acting on site N (the boundary in a chain of N sites) in a suitably chosen representation. T_u satisfies $T_u^N = U$ where $U = \prod_{j=1}^N u_j$ is the element in the symmetry group of H corresponding to the twist at the boundary defined by u. In this letter we consider the existence of such an operator T_u as a definition for toroidal boundary conditions. By the introduction of symmetry-breaking magnetic fields the symmetry of the system will be reduced to a subgroup of $\mathscr{G}$ of order $p' < p$ and correspondingly the number of possible toroidal boundary conditions will decrease to p'.

Such systems have another interesting property if a second-order phase transition occurs for some temperature $1/\lambda$: Up to the non-universal bulk free energy the finite-size scaling spectrum is completely determined by conformal invariance [7]: We denote by E_0 the ground state energy of the model with periodic boundary conditions. Then in the thermodynamic limit $N\to\infty$ the scaled energy gaps $\varepsilon_l=(N/2\pi)(E_l-E_0)=\Delta+\bar{\Delta}+r+\bar{r}$ of the Hamiltonian with any kind of toroidal boundary conditions are given by the highest weights $(\Delta, \bar{\Delta})$ of the irreducible highest weight representations of two commuting Virasoro algebras and some non-negative integers $r, \bar{r}$. The quantities $x=\Delta+\bar{\Delta}$ represent the anomalous dimensions of the fields describing the model at criticality, the quantities $s=\Delta-\bar{\Delta}$ their spin. The multiplicities of the integer-spaced descendant levels with $r, \bar{r}\neq 0$ are given by the character functions of the corresponding highest weight representations [8]†.

It is the aim of this letter to show that the set of possible toroidal boundary conditions is not exhausted by those generated by the global symmetry $\mathscr{G}$ of self-dual Hamiltonians $H(\lambda)$. It turns out that at their self-dual point $\lambda=1$ the duality transformation becomes a true symmetry of the models. As we will show in the n-state Potts models, this additional symmetry allows for a new type of toroidal boundary conditions and we will give explicit representations of H with these 'duality twisted' boundary conditions as well as the representations of the corresponding translation operators. Unlike the symmetry $\mathscr{G}$ which is broken by *magnetic fields*, this symmetry vanishes by changing the *temperature* to $\lambda\neq 1$. In the example of the Ising model we present the complete finite-size scaling spectra for these new boundary conditions as obtained from a projection mechanism in the XXZ Heisenberg chain [9, 10]. The spectra turn out to contain the anomalous dimension of so far unknown spinor operators.

We study the one-dimensional n-state Potts quantum Hamiltonians acting on a chain with N sites

$$H^{(n)}=-\xi^{-1}\left\{\sum_{j=1}^{N-1}\left(e_{2j-1}^{(n)}+\lambda e_{2j}^{(n)}\right)+B_N^{(n)}\right\}. \tag{1}$$

Here ξ^{-1} is a normalization constant fixing the euclidean time scale and λ plays the role of the inverse temperature. The operators $e_l^{(n)}$, $1\leqslant l\leqslant 2N-1$ are given by

$$\begin{aligned} e_{2j-1}^{(n)}&=\frac{1}{n}\sum_{k=1}^{n}\left(\Gamma_j^{(n)}\right)^k \\ e_{2j}^{(n)}&=\frac{1}{n}\sum_{k=1}^{n}\left(\sigma_j^{(n)}\right)^k\left(\sigma_{j+1}^{(n)}\right)^{n-k} \end{aligned} \tag{2}$$

where $\Gamma_j^{(n)}$ and $\sigma_j^{(n)}$ are $n\times n$ matrices acting on site j which satisfy the relations (with $\omega=\exp(2\pi i/n)$)

$$\begin{aligned} &\left(\Gamma_j^{(n)}\right)^n=\left(\sigma_j^{(n)}\right)^n=1 \\ &\left(\Gamma_j^{(n)}\right)^{k\dagger}=\left(\Gamma_j^{(n)}\right)^{n-k} \\ &\left(\sigma_j^{(n)}\right)^{k\dagger}=\left(\sigma_j^{(n)}\right)^{n-k} \\ &\left(\sigma_j^{(n)}\right)^k\left(\Gamma_j^{(n)}\right)^l=\omega^{kl}\left(\Gamma_j^{(n)}\right)^l\left(\sigma_j^{(n)}\right)^k \\ &\left(\sigma_i^{(n)}\right)^k\left(\Gamma_j^{(n)}\right)^l=\left(\Gamma_j^{(n)}\right)^l\left(\sigma_i^{(n)}\right)^k \qquad (i\neq j). \end{aligned} \tag{3}$$

† In the case of certain non-toroidal boundary conditions the spectrum is given by the highest weights of the irreducible representations of only one Virasoro algebra.

$B_N^{(n)}$ is an operator that specifies the boundary conditions (see below). Depending on the boundary conditions the Hamiltonian is symmetric under some subgroup of S_n and therefore splits into various sectors according to the irreducible representations of this group. For periodic boundary conditions the symmetry group is S_n.

Having defined the model we have to discuss the boundary conditions and derive the duality transformation. In what follows, we will omit the superscript (n) in all the quantities defined above. Setting $B_N = e_{2N-1}$ one obtains free boundary conditions. In this case the system is not translationally invariant, but it is important to note that the operators e_j, $1 \leqslant j \leqslant 2N-1$ satisfy the well known Temperley–Lieb algebra, originally introduced by Temperley and Lieb in order to establish relations between the spectra of the Potts quantum chain and the XXZ Heisenberg chain [3, 11]. The Temperley–Lieb algebra with $2N-1$ generators is defined by the relations

$$\begin{aligned} & e_j^2 = e_j \\ & e_j e_{j\pm 1} e_j = \frac{1}{n} e_j \\ & e_j e_k = e_k e_j \qquad \text{if } |k-j| \geqslant 2. \end{aligned} \tag{4}$$

In order to construct the new type of boundary conditions announced above we first discuss the known toroidal boundary conditions arising from the symmetry group S_n by following the procedure discussed in [12]. We define the operators g_j, $(1 \leqslant j \leqslant 2N-1)$ and D by

$$\begin{aligned} & g_j = n\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{n}}\right) e_j - 1 \\ & D = \left(\prod_{j=1}^{2N-1} g_j\right) X \end{aligned} \tag{5}$$

where X is defined by

$$\begin{aligned} & [g_j, X] = 0 \qquad (1 \leqslant j \leqslant 2N-2) \\ & (g_{2N-1} X)^2 = (X g_{2N-1})^2 \end{aligned} \tag{6}$$

and the symbol $\prod$ denotes the ordered product $g_1 g_2 g_3 \ldots$. The operators g_j satisfy $g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}$ (where $1 \leqslant j \leqslant 2N-2$) and $g_j g_k = g_k g_j$ if $|k-j| \geqslant 2$. Together with (6) these are the defining relations for an affine Hecke algebra (see [12] and references therein) and one finds $D e_j = e_{j+1} D$ for $1 \leqslant j \leqslant 2N-2$.

Now we focus on representations of this affine Hecke algebra in which D is invertible and define

$$\underline{e}_{2N} = D e_{2N-1} D^{-1}. \tag{7}$$

The set of operators e_j, $1 \leqslant j \leqslant 2N-1$ and $\underline{e}_{2N}$ satisfies the relations of a periodic Temperley–Lieb algebra [12] with $2N$ generators which is defined by relations (4) together with

$$
\begin{aligned}
&e_{2N}^2 = e_{2N} \\
&e_{2N-1}e_{2N}e_{2N-1} = \frac{1}{n}\, e_{2N-1} \\
&e_{2N}e_{2N-1}e_{2N} = \frac{1}{n}\, e_{2N} \\
&e_{2N}e_1e_{2N} = \frac{1}{n}\, e_{2N} \\
&e_1e_{2N}e_1 = \frac{1}{n}\, e_1 \\
&e_je_{2N} = e_{2N}e_j \qquad \text{if } j \neq 1, 2N-1.
\end{aligned}
\tag{8}
$$

For $n=2$ and $n=3$ (Ising and 3-state Potts models respectively) some classes of solutions to (6) leading to different operators e_{2N} are discussed in [12]. For general n, the simplest solutions we found are of the form $X^{(l)} = \sigma_N^l V$ where $V \in S_n$ is defined by $V^2 = 1$, $V\sigma_j^k V = \sigma_j^{n-k}$, $V\Gamma_j^k V = \Gamma_j^{n-k}$. Note that if a solution X commutes with some element $U \in S_n$, then also XU is a solution to (6). We call such solutions equivalent, since they do not lead to a different e_{2N}.

D acts on the operators e_j, $1 \leqslant j \leqslant 2N-1$, and e_{2N} as follows [12]:

$$
\begin{aligned}
&De_jD^{-1} = e_{j+1} \\
&De_{2N-1}D^{-1} = e_{2N} \\
&De_{2N}D^{-1} = e_1 .
\end{aligned}
\tag{9}
$$

The known types of toroidal boundary conditions are obtained by setting

$$
B_N = e_{2N-1} + \lambda e_{2N} . \tag{10}
$$

This can be seen as follows. By defining $\hat{T} = D^2$ one obtains $\hat{T}e_j\hat{T}^{-1} = e_{j+2}$ $(1 \leqslant j \leqslant 2N-3)$ and similar relations involving e_{2N}. In the representation (2) of e_j this is the definition of the translation operator and one indeed obtains $[H(\lambda), \hat{T}] = 0$ for all values of λ. Furthermore we note that the Nth power of $\hat{T}$ commutes with each of the operators e_j, e_{2N} and therefore must be a linear combination of elements of the symmetry group of H. Thus we conclude that the boundary conditions (10) coincide with the toroidal boundary conditions generated by this group. We call the boundary conditions (10) 'mixed sector' boundary conditions, since one can convince oneself that in the representation (2) of the generators e_j the boundary operator e_{2N} contains a bilocal operator acting on sites N and 1 and (non-local) operators $U \in \mathcal{G}$. As a consequence the boundary conditions depend on the sector (for the Ising model as an example, see below). The various operators $\hat{T}$ obtained from the solutions of (6) are related to the translation operators T_u of the model with specific sector-independent boundary conditions (as defined in the introduction) in such a way that the projection of $\hat{T}$ on some sector of H with boundary condition u coincides with the projection of T_u on this sector.

So far we have shown that D^2 is related to the translation operator. It is easy to see that D is nothing but the duality transformation since from the definition of the Hamiltonian (1) and (10) one finds

$$DH(\lambda)D^{-1}=\lambda H\left(\frac{1}{\lambda}\right). \tag{11}$$

The operators e_{2j} are the dual operators to the operators e_{2j-1}. This leads us to an important observation: At the self-dual point $\lambda=1$ the duality transformation (11) becomes $[H(1), D]=0$, i.e., *D becomes a symmetry operator* of the mixed sector Hamiltonian defined by (1) and (10).

Now we are in a position to construct a Hamiltonian $\tilde{H}$ with new toroidal boundary conditions arising from the additional duality symmetry at the self-dual point. According to the general relationship between a specific toroidal boundary condition and the associated translation operator as discussed in the introduction we would like the translation operator $\tilde{T}$ commuting with the Hamiltonian $\tilde{H}$ to perform a local duality transformation at the boundary (such that $\tilde{T}^N$ performs a global duality transformation D). Studying again a mixed boundary Hamiltonian instead of considering specific sector-independent boundary conditions this means that we require $\tilde{T}e_j\tilde{T}^{-1}=e_{j+2}$, $1\leqslant j\leqslant N-4$, $\tilde{T}e_{2N-3}\tilde{T}^{-1}=\tilde{e}_{2N-1}$., $\tilde{T}e_{2N-2}\tilde{T}^{-1}=e_1$ and $\tilde{T}\tilde{e}_{2N-1}\tilde{T}^{-1}=e_2$ with some operator $\tilde{e}_{2N-1}$.

Repeating the discussion that led from (4) to (9) we realize that these are the relations satisfied by a periodic Temperley–Lieb algebra with $2N-1$ generators and an appropriately defined $\tilde{T}=\tilde{D}^2$. Thus we define a new operator $\tilde{D}$ in analogy to the definition (5) by

$$\tilde{D}=\left(\prod_{j=1}^{2N-2} g_j\right)X \tag{12}$$

with X now being a solution of

$$\begin{aligned} &[g_j, X]=0 \qquad (1\leqslant j\leqslant 2N-3)\\ &(g_{2N-2}X)^2=(Xg_{2N-2})^2. \end{aligned} \tag{13}$$

Solutions to these equations are $X^{(l)}=\Gamma_N^l V$ where V is defined as in the comment after (8). Using the analogy to (9) (one has to replace N by $N-1/2$), we find that $\tilde{H}$ is given by setting

$$B_N=\tilde{e}_{2N-1} \tag{14}$$

with

$$\tilde{e}_{2N-1}=\tilde{D}e_{2N-2}\tilde{D}^{-1}. \tag{15}$$

Obviously the (mixed sector) Hamiltonian $\tilde{H}=H(\lambda=1)$ defined by (1) with boundary term (14) commutes with the duality transformation $\tilde{D}$ defined by (12). Defining the translation operator $\tilde{T}$ by $\tilde{T}=\tilde{D}^2$ one obtains $\tilde{T}^N e_j\tilde{T}^{-N}=\tilde{D}e_j\tilde{D}^{-1}$, i.e., $\tilde{T}^N$ performs a duality transformation as required. As in the case of standard toroidal boundary conditions (9) discussed above the boundary condition B_N defined by (14) contains non-local operators $U\in\mathscr{G}$ and therefore depends on the sector. Note that both the duality and translational invariance break down for $\lambda\neq 1$.

So far we considered mixed sector Hamiltonians. In order to obtain the duality transformation and the translation operator T_u for specific, sector-independent, boundary conditions one has to project on the sectors of the Hamiltonian and express T_u in terms of the various translation operators $\hat{T}$ or $\tilde{T}$ respectively, obtained from the solutions of (6) or (13) respectively. The duality operator $D(\tilde{D})$ does not commute with $H(\tilde{H})$ if the boundary conditions are sector-independent, but rather gives rise to relations of the form $DH_S^B = H_S^B D$ where H_Y^X denotes the projection of the Hamiltonian with boundary condition X on the sector Y. This gives rise to the duality relation given in the introduction for the energy levels.

Let us summarize what we have so far. We first constructed representations of the periodic Temperley–Lieb algebra with $2N$ generators using the duality transformation D and showed that they led to mixed sector versions of Hamiltonians $H(\lambda)$ with toroidal boundary conditions arising from their global symmetry. Then we realized that at the self-dual point $\lambda = 1$ the duality transformation becomes an additional symmetry of $H(1)$ and the toroidal boundary conditions corresponding to this symmetry were shown to be given by the representations of the periodic Temperley–Lieb algebra with $2N-1$ generators. So, in a next step, we constructed these representations in a similar manner as before. This led again to a mixed sector Hamiltonian.

Relating the problem of toroidal boundary conditions to representations of the periodic Temperley–Lieb algebra opens a way of constructing the new boundary operators for sector-independent boundary conditions: One can look directly for representations of the periodic Temperley–Lieb algebra with $2N-1$ generators by choosing the e_j with $1 \leqslant j \leqslant 2N-2$ in the representation (2) and requiring $\tilde{e}_{2N-1}$ to be a hermitian bilocal operator acting on sites N and 1 and to satisfy the analogy to relations (8) with N replaced by $N-1/2$. After some calculation one finds the representations

$$\tilde{e}_{2N-1}^{(l)} = \frac{1}{n} \sum_{k=1}^{n} (A^{(l)})^k \tag{16}$$

where $A = \omega^l \sigma_L^{n-1} \Gamma_L^{2l} \sigma_1$ with $\omega = \exp(2\pi i/n)$ and $1 \leqslant l \leqslant n-1$ if n is odd and $l = \frac{1}{2}, 1, \ldots, (n-1)/2, (n+1)/2, \ldots, n-1, n-\frac{1}{2}$ if n is even respectively. This is an alternative presentation of the results obtained above avoiding the need to work with projection operators. We did not check whether there are other non-equivalent representations of $\tilde{e}_{2N-1}$.

To illustrate our results and to give an explicit application we consider the new boundary conditions in the Ising model. The Ising model is obtained by taking $n = 2$ in (1)–(5). First we illustrate our discussion in the case of the known, sector-independent, boundary conditions. In terms of Pauli matrices one has

$$\begin{aligned} e_{2j-1} &= \tfrac{1}{2}(1 + \sigma_j^x) \\ e_{2j} &= \tfrac{1}{2}(1 + \sigma_j^z \sigma_{j+1}^z). \end{aligned} \tag{17}$$

Periodic boundary conditions are obtained by taking $B_N = e_{2N-1} + \lambda e_{2N}^{(P)}$ with $e_{2N}^{(P)} = \frac{1}{2}(1 + \sigma_N^z \sigma_1^z)$ and we denote the Hamiltonian with periodic boundary conditions by H^0. The global symmetry is S_2, and H^0 commutes with the spin flip operator

$$C = \prod_{j=1}^{N} \sigma_j^x \tag{18}$$

splitting H^0 into two sectors which are even and odd under the action of C. We denote the corresponding projections on these two sectors by H_0^0 and H_1^0. The S_2 symmetry gives rise to one more kind of toroidal boundary conditions which we call antiperiodic boundary conditions obtained by taking $B_N = e_{2N-1} + \lambda e_{2N}^{(A)}$ with $e_{2N}^{(A)} = \frac{1}{2}(1 - \sigma_N^z \sigma_1^z)$. We denote the Hamiltonian with antiperiodic boundary conditions by H^1 and the projections on the even and odd subspaces by H_0^1 and H_1^1 respectively. Obviously H^0 is translationally invariant and commutes with the translation operator

$$T_P = \prod_{j=1}^{N-1} P_{j,j+1} \tag{19}$$

where $P_{j,j+1} = \frac{1}{2}(1 + \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z)$ permutes the spins at two neighbouring sites j and $j+1$. On the other hand H^1 commutes with the generalized translation operator

$$T_A = T_P \cdot \sigma_N^x \tag{20}$$

satisfying $(T_A)^N = C$. This illustrates the discussion in the introduction in the case of sector-independent boundary conditions. Note that the operators e_j, $1 \leqslant j \leqslant 2N-1$ and $e_{2N}^{(P)}$ or $e_{2N}^{(A)}$ respectively verify the relations of the periodic Temperley–Lieb algebra with $2N$ generators.

Now we consider the mixed sector Hamiltonian. There are two non-equivalent solutions to (6), $X^{(0)} = 1$ and $X^{(1)} = \sigma_N^z$ giving rise to two duality operators $D^{(0)}$ and $D^{(1)}$ defined by (5). From them we obtain

$$\begin{aligned} e_{2N}^{(+)} &= \tfrac{1}{2}(1 + C\sigma_N^z \sigma_1^z) \\ e_{2N}^{(-)} &= \tfrac{1}{2}(1 - C\sigma_N^z \sigma_1^z). \end{aligned} \tag{21}$$

Thus taking $B_N = e_{2N-1} + \lambda e_{2N}^{(+)}$ corresponds to periodic boundary conditions in the even sector and antiperiodic boundary conditions in the odd sector while the choice $B_N = e_{2N-1} + \lambda e_{2N}^{(-)}$ corresponds to periodic boundary conditions in the odd sector and antiperiodic boundary conditions in the even sector. The mixed sector Hamiltonian $H^{(\pm)}$ with boundary conditions (21) commutes with the duality transformation $D^{(0,1)}$. This gives rise to the well known duality relations for the projections on the subsectors

$$D^{(Q+Q')} H_Q^{Q'}(\lambda) = \lambda H_{Q'}^{Q}\left(\frac{1}{\lambda}\right) D^{(Q+Q')} \tag{22}$$

with $Q, Q' = 0, 1$ and addition in Q, Q' defined modulo 2.

Defining the projectors $Z^{\pm} = \frac{1}{2}(1 \pm C)$ on the even and odd subsectors and using the relations satisfied by the g_j and X one obtains after some calculation the translation operators

$$\begin{aligned} \hat{T}^{(0)} &= (D^{(0)})^2 = i^{L-1}(T_P Z^+ + T_A Z^-) \\ \hat{T}^{(1)} &= (D^{(1)})^2 = i^L (T_P Z^- + T_A Z^+) \end{aligned} \tag{23}$$

from which in turn T_P and T_A can be obtained in terms of $T^{(0)}$ and $T^{(1)}$ if one is interested in the translation operators for sector-independent boundary conditions.

After this brief review of the known boundary conditions we consider the new 'duality twisted' boundary conditions. There are two non-equivalent solutions to (13), $\tilde{x}_N^{(0)} = 1$ and $\tilde{x}_N^{(1)} = \sigma_N^x$ giving rise to two different duality operators $\tilde{D}^{(0)}$ and $\tilde{D}^{(1)}$ defined by (12) and to two new boundary operators

$$\begin{aligned}\tilde{e}^{(+)}_{2N-1} &= \tfrac{1}{2}(1 - C\sigma^y_N\sigma^-_1)\\ \tilde{e}^{(-)}_{2N-1} &= \tfrac{1}{2}(1 + C\sigma^y_N\sigma^-_1)\end{aligned} \tag{24}$$

where $\tilde{e}^{(-)}_{2N-1}$ is the complex conjugate of $\tilde{e}^{(+)}_{2N-1}$. Note that as opposed to the generalized boundary conditions discussed in [13] the Hamiltonian does *not* contain the operator σ^x_L.

From the duality operators $\tilde{D}^{(Q)}$ we obtain after some calculation the translation operators

$$\begin{aligned}\tilde{T}^{(0)} &= (\tilde{D}^{(0)})^2 = i^L T_P(Z^+ d^*_L - iZ^- d_L)\\ \tilde{T}^{(1)} &= (\tilde{D}^{(1)})^2 = (-i)^L T_P(Z^+ d_L + iZ^- d^*_L)\end{aligned} \tag{25}$$

with $d_L = g_{2L-2}g_{2L-1}$. In this expression both g_{2L-2} and g_{2L-1} are defined by (5) with the corresponding e_j in the representation (17). The asterisk marks complex conjugation.

The corresponding sector-independent boundary operators satisfying the relations of the periodic Temperley–Lieb algebra are given by

$$\tilde{e}_{2N-1} = \tfrac{1}{2}(1 + \sigma^y_N\sigma^-_1) \tag{26}$$

and its complex conjugate. The translation operator commuting with $\tilde{H}$ with this boundary condition is obtained from (25) through projection on the even and odd sectors and given by

$$T_D = i^{-L}(\tilde{T}^{(1)}Z^+ + i\tilde{T}^{(0)}Z^-) = T_P d_L \tag{27}$$

and its complex conjugate.

Finally we discuss the spectrum of $\tilde{H}$. Since the Ising Hamiltonian with boundary condition $B_N = \tilde{e}_{2N-1}$ (26) is hermitian, the spectrum with the complex conjugate boundary condition B^*_N is identical. In addition to the symmetries discussed above, $\tilde{H}$ commutes with the operator

$$\tilde{P} = \left(\prod_{j=1}^{[(N+1)/2-1]} P_{j,N-j}\right)(\cos(\pi/4)\sigma^x_N - \sin(\pi/4)\sigma^y_N) \tag{28}$$

where $P_{j,j'}$ is the permutation operator defined above and $[L]$ denotes the integer part of the number L. $\tilde{P}$ does not commute with the spin flip operator C and therefore the two sectors of $\tilde{H}$ are degenerate.

The finite-size scaling spectra for the two (degenerate) sectors can be obtained from the XXZ Heisenberg chain with an odd number of sites [10]. As discussed in the introduction, the scaled energy gaps $\mathscr{E}_l$ in the thermodynamic limit are given by the irreducible highest weight representations $(\Delta, \bar{\Delta})$ of the Virasoro algebra with central charge $c = 1/2$ describing the critical Ising model. Therefore we denote the scaled energy gaps $\mathscr{E}$ by the various representations $(\Delta, \bar{\Delta})$ contributing to them and obtain for each sector [10]

$$\mathscr{E} = (0, \tfrac{1}{16}) + (\tfrac{1}{2}, \tfrac{1}{16}). \tag{29}$$

For the complex conjugate boundary conditions one obtains

$$\mathscr{E} = (\tfrac{1}{16}, 0) + (\tfrac{1}{16}, \tfrac{1}{2}). \tag{30}$$

To the best of our knowledge operators with anomalous dimensions $x=1/16$ and $x=9/16$ and spin $s=\pm 1/16$ and $s=\pm 7/16$ respectively, have not been discussed in connection with the critical Ising model.

To conclude let us note that the finite-size scaling spectra of the 3-states Potts model with boundary conditions given by (15) or (16) respectively, are also explicitly known from the projection mechanism in the XXZ Heisenberg chain [10]. It would be very interesting to study the possibility of 'duality twisted' boundary conditions in other critical systems. We have left open the question of completeness of the solutions to (13) and of the representations (16) respectively. This problem is addressed in [12] for the periodic Temperley–Lieb algebra with an even number of generators and needs further investigation, in particular in the case of an odd number of generators.

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